

The generalized Pillai equation $\pm ra^x \pm sb^y = c$

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Abstract

In this paper we consider N , the number of solutions (x, y, u, v) to the equation $(-1)^u ra^x + (-1)^v sb^y = c$ in nonnegative integers x, y and integers $u, v \in \{0, 1\}$, for given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$. We show that $N \leq 2$ when $\gcd(ra, sb) = 1$ and $\min(x, y) > 0$, except for a finite number of cases that can be found in a finite number of steps. For arbitrary $\gcd(ra, sb)$ and $\min(x, y) \geq 0$, we show that when $(u, v) = (0, 1)$ we have $N \leq 3$, with an infinite number of cases for which $N = 3$.

1 Introduction

In this paper we consider N , the number of solutions (x, y, u, v) to the equation

$$(-1)^u ra^x + (-1)^v sb^y = c \tag{P}$$

in nonnegative integers x, y and integers $u, v \in \{0, 1\}$, for given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$.

Equation (P) is a generalized form of the familiar Pillai equation $ra^x - sb^y = c$, also referred to as Pillai's Diophantine equation as in [4] and [21]. This equation has been considered by many authors from a variety of standpoints; several subsets of the set $\{a, b, r, s, x, y\}$ have been treated as the unknowns. Here we will consider only the case in which a , b , r , s , and c are given. For histories of other cases, as well as a more detailed history of the case under consideration here, we refer the reader to the surveys [3] and [21].

The result that N is finite follows easily from an effective result of Ellison [5], which is a response to an earlier result of Pillai [12]. Recent work has focused on finding small upper bounds on N .

The case $rs = 1$ has received particular attention. Le [9] showed that for $rs = 1$, $N \leq 2$ under the restrictions $x > 1$, $y > 1$, $a \geq 10^5$, $b \geq 10^5$, $u = 0$, $v = 1$, and $(a, b) = 1$. Bennett [1] relaxed these restrictions to $x > 0$, $y > 0$, $a > 1$, $b > 1$, and later the authors [17] removed the restrictions on u and v , showing that $N \leq 2$ whenever $rs = 1$ and $\min(x, y) > 0$, except for listed exceptional cases. The results of [1] and [17] also remove the restriction $(a, b) = 1$.

The case $rs > 1$ was treated by Shorey [18], who showed that Equation (P) has at most 9 solutions in positive integers (x, y) when $(u, v) = (0, 1)$ and the terms on the left side of (P) are large relative to c . Later, under the restrictions $x > 1$, $y > 1$, and $(ra, sb) = 1$, Le [9] obtained the following improved value for N :

Theorem L [9] If $u = 0$, $v = 1$, $x > 1$, $y > 1$, $a \geq e^e$, $b \geq e^e$, and $(ra, sb) = 1$, then $N \leq 3$.

In this paper we remove the restrictions $x > 1$, $y > 1$, $a \geq e^e$, $b \geq e^e$, and $(ra, sb) = 1$ (see Theorem 3 in Section 3). When $(ra, sb) = 1$ and $\min(x, y) > 0$, we show that $N \leq 2$, even when u and v are unrestricted, except for a finite number of cases that can be found in a finite number of steps (see Theorem 1 in Section 2.) The second author searched the ranges $1 < b < a \leq 100$, $1 \leq r, s \leq 1000$, $(ra, sb) = 1$, $a \nmid r$, $b \nmid s$, a and b not perfect powers, looking for two solutions (x, y) satisfying $1 \leq x_1, x_2 \leq 12$ and $1 \leq y_1, y_2 \leq 12$, and then looking for a third solution satisfying $x_3 \leq 24$ and $y_3 \leq 24$; the search found no cases of $N \geq 3$ other

than $(a, b, c, r, s) = (3, 2, 1, 1, 1), (3, 2, 5, 1, 1), (3, 2, 7, 1, 1), (3, 2, 11, 1, 1), (3, 2, 13, 1, 1)$, and $(5, 2, 3, 1, 1)$. We note that these exceptional (a, b, c, r, s) all have $r = s = 1$ and a and b both primes, so that they correspond to the list of exceptional cases in the elementary Theorem 7 of [16].

Theorem 1 uses a result of Matveev [10], and Theorem 3 uses a recent result of He and Togbé [8].

2 $\gcd(ra, sb) = 1$ and $\min(x, y) > 0$, with u and v unrestricted

Theorem 1. *Let $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$ be positive integers with $(ra, sb) = 1$. Then the equation*

$$(-1)^u ra^x + (-1)^v sb^y = c \quad (1)$$

has at most two solutions (x, y, u, v) in positive integers x, y and integers $u, v \in \{0, 1\}$, except for a finite number of cases which can be found in a finite number of steps. More specifically, if (1) has more than two solutions, then $\max(a, b, r, s, x, y) < 8 \cdot 10^{14}$ for each solution.

Corollary to Theorem 1. *Let $1 < b < a \leq 15$, $c > 0$, $1 \leq r, s \leq 100$ be positive integers with $(ra, sb) = 1$. Then (1) has at most two solutions (x, y, u, v) in positive integers x, y and integers $u, v \in \{0, 1\}$, unless $(a, b, c, r, s) = (3, 2, 1, 1, 1), (3, 2, 5, 1, 1), (3, 2, 5, 1, 2), (3, 2, 7, 1, 1), (3, 2, 11, 1, 1), (3, 2, 13, 1, 1), (3, 2, 13, 1, 2), (4, 3, 13, 1, 1)$, or $(5, 2, 3, 1, 1)$.*

Unlike the computer search mentioned in the introduction, the Corollary to Theorem 1 does not use the restrictions $a \nmid r$, $b \nmid s$, a and b not perfect powers.

Throughout this section, we take $(ra, sb) = 1$ and $\min(x, y) > 0$.

The proofs of Theorem 1 and its Corollary will follow several lemmas needed to prepare for the proof of Theorem 1.

Since for each choice of (x, y) giving a solution to (1), (u, v) is determined, we will usually refer simply to a solution (x, y) .

Lemma 1. *Let $a > 1$, $b > 1$, $r > 0$, and $m > 0$ be integers. Suppose there exist positive integers y such that $(b^y \pm 1)/(ra^m)$ is an integer prime to a , and let n be the least such y possible regardless of the choice of sign. Then if $M > m$ and*

$$b^N \pm 1 = ra^M L$$

where the \pm is independent of the above and L and a are not necessarily relatively prime, we must have

$$n \frac{a^{M-m}}{2^{g+h-1}} \mid N,$$

where $g = 1$ and $h = 0$ unless r is odd, $a \equiv 2 \pmod{4}$, and $m = 1$, in which case g is the largest integer such that $2^g | b \pm 1$ (where the \pm is chosen to maximize g), and h is the largest integer such that $2^h | n$.

Proof. This lemma is essentially Lemma 1 of [17] with a slight generalization, and can be proven in essentially the same way. \square

Lemma 2. *If (1) has two solutions, then, if the value of x is the same in both solutions, there is no third solution except when $(a, b, c, r, s) = (3, 2, 1, 1, 1), (3, 2, 5, 1, 1), (5, 2, 3, 1, 1)$, or $(3, 2, 7, 1, 1)$.*

Proof. Suppose (1) has two solutions, (x_1, y_1) and (x_2, y_2) with $x_1 = x_2$. We can take $y_1 < y_2$. Then

$$2ra^{x_1} = sb^{y_1}(b^h \pm 1) \quad (2)$$

where $h = y_2 - y_1 > 0$. We see that $b = 2$, $s = y_1 = 1$, $ra^{x_1} = 2^h \pm 1$ and $c = 2^h \mp 1$. If $h \leq 2$, then (a, b, c, r, s) must be one of the first three exceptional cases in the formulation of the lemma. So we take $h > 2$. Now suppose (1) has a third solution (x_3, y_3) . If $x_3 = x_1$, then (x_3, y_3) must be a duplicate solution. If $x_3 < x_1$ then we have

$$ra^{x_3}(a^{x_1-x_3} \pm 1) = 2(2^{y_3-1} \pm 1) \quad (3)$$

where the \pm signs are independent of each other and also independent of the sign in (2). From (3) it follows that there exists a least number n such that

$$2^n \pm 1 = ra^{x_3}l \quad (4)$$

where $(a, l) = 1$ and the sign in (4) is independent of the signs in (2) and (3). By Lemma 1, there must be a prime $p|a$ such that $np|h$. Also, $n | h$ implies $ra^{x_3}l | ra^{x_1}$ (even when the sign in (4) does not match the sign in (2)) so that $l = 1$. Now

$$2^{np} \pm 1 = ra^{x_3}pl_1 \quad (5)$$

where $(l_1, ra) = 1$ and the \pm agrees with (4). If $ra^{x_3} \neq 3$, we must have $l_1 > 1$, impossible since $np | h$ implies $ra^{x_3}pl_1 | ra^{x_1}$ (even if the sign in (5) does not match the sign in (2)). So we must have $ra^{x_3} = 3$, forcing $2^h \pm 1 = 9$, yielding the fourth exceptional case in the formulation of the lemma.

So we can assume $x_3 > x_1$. Now

$$ra^{x_3} = \pm 2^{y_3} \pm c \quad (6)$$

where the \pm signs are independent. Suppose $y_3 \leq h$. Then

$$ra^{x_3} \geq 3ra^{x_1} \geq 3(2^h - 1) > 2^h + (2^h + 1) \geq 2^{y_3} + c$$

contradicting (6). So $y_3 > h$. Letting $a^{x_3-x_1} = 2^{kt} \pm 1$ with t odd and $k \geq 2$, and considering (6) modulo 2^{h+1} , we see that $k > h$ so that $x_3 - x_1$ is even (note $a \leq 2^h + 1 < 2^{h+1} - 1$). So $a^{x_3-x_1} \equiv 1 \pmod{8}$, and $ra^{x_1} \equiv -c \pmod{8}$. Now considering (6) modulo 8, we see that we must have

$$ra^{x_3} + c = 2^{y_3}$$

since no other combination of signs is possible. But since we also have $ra^{x_1} + c = 2^{y_2}$ and $x_3 - x_1$ is even, we have a contradiction to Theorem 1 of [15] since $c = 2^h \mp 1 \neq 3, 5, \text{ or } 13$. \square

Lemma 3. *If (1) has two solutions (x_1, y_1) and (x_2, y_2) with $x_1 \leq x_2$, we must have $ra^{x_2} > c/2$. If in addition there is a third solution (x_3, y_3) with $x_3 \geq x_2$ we must have $ra^{x_3} > c$.*

Proof. Assume (1) has two solutions (x_1, y_1) and (x_2, y_2) with $x_1 \leq x_2$. Since $(ra, sb) = 1$, c is prime to $rasb$. If $ra^{x_2} < c/2$, then $sb^{y_2} > c/2$ by (1), and also $ra^{x_1} < c/2$ so that $sb^{y_1} > c/2$. But we have

$$ra^{x_1}(a^{x_2-x_1} \pm 1) = sb^{\min(y_1, y_2)}(b^{|y_2-y_1|} \pm 1)$$

where the \pm signs are independent, so that

$$c/2 > ra^{x_2} \geq a^{x_2-x_1} + 1 \geq sb^{\min(y_1, y_2)} > c/2,$$

a contradiction. If there is a further solution (x_3, y_3) with $x_3 \geq x_2$, then, by Lemma 2, we can take $x_1 < x_2 < x_3$, so that

$$ra^{x_3} \geq 2ra^{x_2} > c,$$

unless (a, b, c, r, s) is one of the exceptional cases of Lemma 2, in which case it suffices to observe that we must have $a^{x_3} \geq a^2 \geq 9 > c$. \square

Lemma 4. *If (1) has three solutions (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) then*

$$c < (ZJ)^2$$

where $J = \max(a, b)$ and $Z = \max(x_1, x_2, x_3, y_1, y_2, y_3)$.

Proof. Since $(ra, sb) = 1$, we can take sb odd without loss of generality. Assume (1) has three solutions (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . By Lemma 2 (noting that, for the exceptional cases of Lemma 2, $(ZJ)^2 > 9 > c$), we can take $x_1 < x_2 < x_3$ and we can further assume no two of y_1 , y_2 , and y_3 are equal. Write $c^{1/t} = sb^{y_0}$ where $y_0 = \min(y_1, y_2, y_3)$ and t is some real number. Let $i, j \in \{1, 2, 3\}$ with $i < j$. Then

$$ra^{x_i}(a^{x_j-x_i} \pm 1) = sb^{\min(y_i, y_j)}(b^{|y_j-y_i|} \pm 1) \quad (7)$$

where the \pm signs are independent. Let n be the least positive integer such that $(b^n \pm 1)/ra^{x_1}$ is an integer prime to a , where the \pm sign is independent of (7) (that such n exists follows from (7)). From (7) we can apply Lemma 1 (with g and h defined as in Lemma 1) to get

$$Z > |y_3 - y_2| \geq n \frac{a^{x_2-x_1}}{2^{g+h-1}} \geq \frac{a^{x_2-x_1}}{2^{g-1}} \geq \frac{sb^{y_0} - 1}{(b+1)/2} = \frac{c^{1/t} - 1}{(b+1)/2} \geq \frac{c^{1/t}}{b}$$

since $b \geq 3$. Thus,

$$c < (bZ)^t. \quad (8)$$

Now reorder the three solutions (x, y) as (X_1, Y_1) , (X_2, Y_2) , (X_3, Y_3) so that $Y_1 < Y_2 < Y_3$. Now, again using Lemma 1, we have

$$Z > |X_3 - X_2| \geq b^{Y_2-Y_1} = \frac{sb^{Y_2}}{sb^{Y_1}} > \frac{c/2}{c^{1/t}}$$

where the last inequality follows from Lemma 3 (with the roles of a and b reversed). So now

$$c < (2Z)^{\frac{t}{t-1}}. \quad (9)$$

Using (8) if $t \leq 2$ and using (9) if $t > 2$, we see that the lemma holds. \square

Lemma 5. *If (1) has three solutions (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , then $\max(x_1, y_1, x_2, y_2, x_3, y_3) \geq \max(r, s, a, b)$.*

Proof. Assume (1) has three solutions (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . By Lemma 2 (noting that for the exceptional cases of Lemma 2 we must have $y_3 \geq a = \max(r, s, a, b)$), we can take $x_1 < x_2 < x_3$ and can further assume no two of y_1 , y_2 , and y_3 are equal. Let n be the least positive integer such that $(b^n \pm 1)/ra^{x_1}$ is an integer prime to a (that such n exists follows from (7)). Let $Z = \max(x_1, y_1, x_2, y_2, x_3, y_3)$. We can apply Lemma 1 and (7) to get

$$Z > |y_3 - y_2| \geq n \frac{a^{x_2-x_1}}{2^{g+h-1}} \geq \frac{a^{x_2-x_1}}{2^{g-1}} \geq \frac{sb-1}{2^{g-1}},$$

where g and h are as in Lemma 1. If a is odd, then $g = 1$ and

$$Z \geq \max(a, b, s).$$

If a is even, then, using $2^{g-1} \leq (b+1)/2$ and $b \geq 3$, we get

$$Z > \frac{sb-1}{(b+1)/2} \geq s.$$

In the same manner, reversing the roles of a and b , we get, when b is odd,

$$Z \geq \max(a, b, r),$$

and, when b is even, we get

$$Z > r.$$

Since $(ra, sb) = 1$, at least one of a or b must be odd, so that the lemma holds. \square

Lemma 6. (Matveev [10] as given in [11, Theorem 1]) *Let $\lambda_1, \lambda_2, \lambda_3$ be \mathbb{Q} -linearly independent logarithms of non-zero algebraic numbers and let b_1, b_2, b_3 be rational integers with $b_1 \neq 0$. Define $\alpha_j = \exp(\lambda_j)$ for $j = 1, 2, 3$ and*

$$\Lambda = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over \mathbb{Q} . Put

$$\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

Let A_1, A_2, A_3 be positive real numbers, which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\lambda_j|, 0.16\} \quad (1 \leq j \leq 3).$$

Assume that

$$B \geq \max\{1, \max\{|b_j|A_j/A_1 : 1 \leq j \leq 3\}\}.$$

Define also

$$C_1 = \frac{5 \times 16^5}{6\chi} e^3 (7 + 2\chi) \left(\frac{3e}{2}\right)^\chi \left(20.2 + \log(3^{5.5} D^2 \log(eD))\right).$$

Then

$$\log |\Lambda| > -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)).$$

Proof of Theorem 1. Recall that, for a given solution, (x, y) uniquely determines (u, v) , so we refer to a solution (x, y) . Assume (1) has three solutions (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . Let $Z = \max(x_1, y_1, x_2, y_2, x_3, y_3)$. Without loss of generality, we can take $Z = \max(x_3, y_3)$. Also let $D = \max(ra^{x_3}, sb^{y_3})$, $d = \min(ra^{x_3}, sb^{y_3})$, $J = \max(a, b)$, and $j = \min(a, b)$. By Lemma 3, $D > c$, so that

$$|ra^{x_3} - sb^{y_3}| = D - d = c.$$

Let

$$\Lambda = |\log(r/s) + x_3 \log(a) - y_3 \log(b)| = \log(D) - \log(d) = \log(1 + c/d) < c/d$$

so that

$$\log(1/\Lambda) > \log(d) - \log(c). \quad (10)$$

Theorem 1 of [17], along with the list of solutions in Theorem 6 of [17], handles the case $rs = 1$ and, therefore, also handles the case in which both r and a are powers of the same base and b and s are powers of the same base, so from here on we assume $rs > 1$ and also assume that there do not exist nonnegative integers h, k, w_1, w_2, z_1 , and z_2 such that $r = h^{w_1}$, $a = h^{w_2}$, $s = k^{z_1}$, and $b = k^{z_2}$.

Now we can apply Lemma 6 with $(\alpha_1, \alpha_2, \alpha_3) = (r/s, a, b)$, $(A_1, A_2, A_3) = (\log(\max(r, s)), \log(a), \log(b))$, and $B = Z$ to get

$$\log(1/\Lambda) < CA_1A_2A_3 \log(1.5eB), \quad (11)$$

where $C = 1.6901816335 \cdot 10^{10}$. Combining (10) and (11) we get

$$\log(d) < \log(c) + C \log(\max(r, s)) \log(a) \log(b) \log(1.5eZ). \quad (12)$$

Also $\Lambda = \log(1 + c/d) \leq \max(\log(c), \log(2))$, so that, adding Λ to both sides of (12), we get

$$\log(D) < \max(\log(c), \log(2)) + \log(c) + C \log(\max(r, s)) \log(a) \log(b) \log(1.5eZ). \quad (13)$$

Using (13) and noting that $Z \log(j) \leq \log(D)$, we have

$$Z \log(j) < \max(\log(c), \log(2)) + \log(c) + C \log(\max(r, s)) \log(a) \log(b) \log(1.5eZ).$$

Dividing through by $\log(j)$ and noting $j \geq 2$, we get

$$Z < \frac{\max(\log(c), \log(2)) + \log(c)}{\log(2)} + C \log(\max(r, s)) \log(J) \log(1.5eZ). \quad (14)$$

Now applying Lemma 4, we have $c < (ZJ)^2$, so that (14) gives

$$Z < \frac{4 \log(ZJ)}{\log(2)} + C \log(\max(r, s)) \log(J) \log(1.5eZ).$$

Now applying Lemma 5 we have

$$Z < \frac{8 \log(Z)}{\log(2)} + C(\log(Z))^2 \log(4.078Z).$$

From this we obtain $Z < 8 \cdot 10^{14}$. Applying Lemma 5 again proves the theorem. \square

Proof of Corollary to Theorem 1: The exceptions listed in the formulation of the corollary are all given by Theorem 1 of [17], which handles the case $rs = 1$ and, therefore, handles the case in which r and a are both powers of the same base and b and s are both powers of the same base, so from here on we exclude these cases from consideration. Since any solution to (1) can be rewritten as a case in which $a \nmid r$, $b \nmid s$, and a and b are not perfect powers, we use these restrictions in proceeding with a computer search.

Suppose we have three solutions (x_k, y_k, u_k, v_k) with $k = 1, 2$, and 3 . For $i, j \in \{1, 2, 3\}$, we rewrite $(-1)^{u_i} r a^{x_i} + (-1)^{v_i} s b^{y_i} = c = (-1)^{u_j} r a^{x_j} + (-1)^{v_j} s b^{y_j}$ as

$$r a^{x_0} (a^{x_h - x_0} + (-1)^m) = s b^{y_0} (b^{y_h - y_0} + (-1)^n) \quad (15)$$

where $x_0 = \min(x_i, x_j)$, $y_0 = \min(y_i, y_j)$, $x_h = \max(x_i, x_j)$, $y_h = \max(y_i, y_j)$, and m and n are in $\{0, 1\}$.

For each choice of (r, a, s, b) we use the technique known as ‘bootstrapping’ (see [7] and [19]) to find increasingly stringent congruence conditions on the exponents $x_h - x_0$ and $y_h - y_0$. When these conditions show that either $x_h - x_0$ or $y_h - y_0$ exceeds $8 \cdot 10^{14}$, by Theorem 1 there can be no third solution. (The bootstrapping in [7] and [19] deals only with the case $m = n = 1$ but the ideas extend easily to the other choices of signs, indeed, more easily since for $m = 0$ or $n = 0$ parity considerations often lead to a contradiction, as in the proof of Lemma 9 of [17]. The Maple programs used can be found on the second author’s website [20].) Since we have three solutions to (1), Lemma 2 shows that for at least one pair of solutions we have $x_0 \geq 2$ (alternatively, $y_0 \geq 2$). For a given (r, a, s, b) , we treat each pair (m, n) : when $(m, n) = (0, 0)$, $(0, 1)$, or $(1, 0)$, we take first $x_0 \geq 2$ and $y_0 \geq 1$, and then take $x_0 \geq 1$ and $y_0 \geq 2$, and use the bootstrapping method to find those few cases allowing two solutions; when $m = n = 1$, we take $x_0 \geq 2$ and $y_0 \geq 2$ and bootstrap in the same way (to see that these lower bounds on x_2 and y_2 are justified, we note the following: if (x_0, y_0) is a solution to (1) then Lemma 2 shows that we can get (15) with $\min(x_0, y_0) \geq 2$. If (x_0, y_0) is not a solution to (1), then it is not hard to see that we cannot have any two different solutions to (15) both having $m = n = 1$ since then all three solutions to (1) would have the same (u, v) which is impossible unless (x_0, y_0) is a solution to (1); so we can assume that in this case at least one solution to (15) has $(m, n) \neq (1, 1)$ with $\max(x_0, y_0) \geq 2$.) In the ranges under consideration, we find only a small number of choices of (r, a, s, b) for which (1) could have three solutions.

Choose one such (r, a, s, b) ; for each of the four possible choices of (m, n) , we use the bootstrapping method again to find numbers k_x and k_y such that if either $x_0 > k_x$ or $y_0 > k_y$ then (15) has no solution. For each (m, n) , we take each pair (x_0, y_0) such that $x_0 \leq k_x$ and $y_0 \leq k_y$; either bootstrapping finds a solution $(x_0, y_0, x_h, y_h, m, n)$ to (15) with the minimal x_h and y_h , or bootstrapping shows no solution exists for that (x_0, y_0) . In this way we obtain a complete list of quadruples (x_0, y_0, m, n) that could occur in a solution of (15), for each of which we have found one solution $(x_0, y_0, x_h, y_h, m, n)$. Suppose one such quadruple admits a second solution $(x_0, y_0, x_H, y_H, m, n)$. Then we must have

$$ra^{x_h} - sb^{y_h} = ra^{x_H} - sb^{y_H},$$

with $x_h < x_H$ and $y_h < y_H$, so that our list of quadruples must include $(x_h, y_h, 1, 1)$. We easily check that such $(x_h, y_h, 1, 1)$ does not appear as one of the listed quadruples. Thus, each quadruple on the list has only one solution, so that the associated solutions $(x_0, y_0, x_h, y_h, m, n)$ give all solutions to (15). For each such solution, there are two possible values of c . We compute all these c values and find that no two are equal, confirming the corollary for this choice of (r, a, s, b) . Continuing in this way for each (r, a, s, b) completes the proof of the corollary. \square

The following theorem simply observes that cases of exactly two solutions to (1) are commonplace and easy to construct.

Theorem 2. *There are an infinite number of cases of exactly two solutions to (1). More specifically, for every choice of (a, b) , where a and b are relatively prime and not perfect powers, there are an infinite number of choices of (x_1, y_1) such that for every quadruple (a, b, x_1, y_1) there are an infinite number of quintuples (r, s, c, x_2, y_2) with $(ra, sb) = (r, a) = (s, b) = 1$ such that both (x_1, y_1) and (x_2, y_2) give solutions to (1), and no further solution exists.*

Proof. We define $m(a, b)$ as the least value of $m > 1$ such that $b^n \pm 1 = a^m l$ for some integer n and for some integer l such that $\gcd(l, a) = 1$. As pointed out in [1], such $m(a, b)$ exists by, e.g., Ribenboim [14, C6.5]. (The definition here differs slightly from that in [1].)

For any choice of (a, b) such that $(a, b) = 1$ and a and b are not perfect powers, choose $x_1 \geq m(a, b)$ and $y_1 \geq m(b, a)$. Then it is easily seen that we can find infinitely many choices of r, s, x_2, y_2 , with $(ra, sb) = (r, a) = (s, b) = 1$, such that

$$ra^{x_1}(a^{x_2-x_1} \pm 1) = sb^{y_1}(b^{y_2-y_1} \pm 1). \quad (16)$$

By Theorem 1, we can take x_2 sufficiently high to ensure no third solution exists. \square

3 $(u, v) = (0, 1)$, with $\gcd(ra, sb)$ unrestricted and $\min(x, y) \geq 0$

Let $a > 1, b > 1, c > 0, r > 0$ and $s > 0$ be positive integers. In this section we give a result on the number of solutions in nonnegative integers (x, y) to the equation

$$ra^x - sb^y = c. \quad (17)$$

Throughout this section we allow $(ra, sb) > 1$ and $\min(x, y) = 0$. When $(ra, sb) = 1$ and $\min(x, y) > 0$, note that (17) is (1) with $(u, v) = (0, 1)$.

To eliminate cases of multiple solutions to (17) which are directly derived from other cases by multiplying each term of each solution by a fixed positive number k , we use the following definition: let (x_1, y_1) be the least of N solutions to (17); then, if there exists a positive integer $k > 1$ such that $ra^{x_1}/k = r_1a^w$ and $sb^{y_1}/k = s_1b^z$ for positive integers r_1, s_1 and nonnegative integers w, z , the set of N solutions is called *reducible*. (Each irreducible set of solutions corresponds to an infinite number of reducible sets of solutions.)

We also want to eliminate from consideration cases in which $a \mid r$ or $b \mid s$; we call such solutions *improper*.

Finally, we want to eliminate from consideration cases where a or b is a perfect power; we call such solutions *redundant*.

Just as it is easy to construct cases of exactly two solutions to (1) (see Theorem 2 above), it is easy to construct cases of exactly two solutions to (17), where the pair of solutions (x_1, y_1) and (x_2, y_2) is not reducible, improper, or redundant, since we can obtain $(ra, sb) = (r, a) = (s, b) = 1$ as in (16). (The solutions obtained in this way will have $\min(x, y) > 0$.) Note that in (16) we can always choose the signs on both the right and the left to be minus (when a is even this may require taking $x_1 \geq m(a, b) + 1$; similarly for b even).

We now consider cases of more than two solutions to (17).

Theorem 3. *There are at most three solutions in nonnegative integers (x, y) to (17). There are an infinite number of cases of (17) with three solutions, even if we exclude reducible sets of solutions and also exclude solutions which are improper or redundant.*

Proof. Suppose (17) has more than two solutions and let the two least solutions be (x_1, y_1) and (x_2, y_2) . Let (x_3, y_3) be a third solution, taking $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$. Let $R = \frac{ra^{x_1}}{\gcd(ra^{x_1}, sb^{y_1})}$ and $S = \frac{sb^{y_1}}{\gcd(ra^{x_1}, sb^{y_1})}$. We have

$$R(a^{x_2-x_1} - 1) = S(b^{y_2-y_1} - 1)$$

and

$$R(a^{x_3-x_1} - 1) = S(b^{y_3-y_1} - 1).$$

Let

$$t = \frac{a^{x_2-x_1} - 1}{S} = \frac{b^{y_2-y_1} - 1}{R}$$

and

$$T = \frac{a^{x_3-x_1} - 1}{S} = \frac{b^{y_3-y_1} - 1}{R}.$$

Note that t and T are both integers.

Let $g_1 = \gcd(x_2 - x_1, x_3 - x_1)$ and $g_2 = \gcd(y_2 - y_1, y_3 - y_1)$. Let k be the least integer such that $b^k - 1$ is divisible by R . Then k must divide both $y_2 - y_1$ and $y_3 - y_1$, so that k divides g_2 , and

$$b^{g_2} - 1 = Rl_2$$

for some integer l_2 . Similarly,

$$a^{g_1} - 1 = Sl_1$$

for some integer l_1 . Since g_1 divides both $x_2 - x_1$ and $x_3 - x_1$, l_1 divides t and T . There must be an integer j which is the least positive integer such that $b^j - 1$ is divisible by Rl_1 , and j must divide both $y_2 - y_1$ and $y_3 - y_1$, so that j divides g_2 . Therefore, $l_1 | l_2$.

A similar argument with the roles of a and b reversed shows that $l_2 | l_1$, so that $l_1 = l_2$, and we have

$$ra^{x_1}(a^{g_1} - 1) = sb^{y_1}(b^{g_2} - 1). \quad (18)$$

(18) shows that $(x_1 + g_1, y_1 + g_2)$ is a solution to (17). If $x_1 + g_1 \neq x_2$, then, by the definition of x_2 , we must have $x_1 + g_1 > x_2$, which is impossible by the definition of g_1 . So $x_1 + g_1 = x_2$ and, similarly, $y_1 + g_2 = y_2$. Letting $A = a^{x_2-x_1}$, $B = b^{y_2-y_1}$, $m = \frac{x_3-x_1}{x_2-x_1}$ and $n = \frac{y_3-y_1}{y_2-y_1}$, we find that we have a solution to the Goormaghtigh equation

$$\frac{A^m - 1}{A - 1} = \frac{B^n - 1}{B - 1} \quad (19)$$

in integers $A > 1$, $B > 1$, $m > 1$, $n > 1$. Note that both sides of (19) equal the integer T/t . By the same argument as above, if there exists a fourth solution (x_4, y_4) , we have a solution to (19) for the same values of A and B but with $m = \frac{x_4-x_1}{x_2-x_1}$ and $n = \frac{y_4-y_1}{y_2-y_1}$, contradicting Theorem 1.3 of [8], which states that, for given A and B , (19) has at most one solution (m, n) .

Finally, fixing $n = 2$ and $m > 2$, there are an infinite number of solutions (A, B, m) to (19) each of which corresponds to a set of three solutions to (17) in which

$$(a, b, c, r, s, x_1, y_1, x_2, y_2, x_3, y_3) = (a_0, dA, A(d-1)/h, (dA-1)/h, (A-1)/h, 0, 0, j, 1, mj, 2) \quad (20)$$

where $d = \frac{A^{m-1}-1}{A-1}$, $h = \gcd(dA-1, A-1)$, and $a_0^j = A$ with a_0 not a perfect power. Suppose $b = dA$ is a perfect power. Then there must be a prime $q | j$ such that $d = d_0^q$ and $A = a_1^q$ for some d_0 and $a_1 = a_0^{j/q}$. This gives $d_0^q = \frac{a_1^{q(m-1)}-1}{a_1^q-1}$, requiring $m > 3$. But this contradicts Corollary 1.2(b) of Bennett [2], which states that the equation $(x^n - 1)/(x - 1) = y^q$ in integers $x > 1$, $y > 1$, $n > 2$, $q \geq 2$ has no solution (x, y, n, q) where x is a q -th power. Thus, $b = dA$ is not a perfect power, so that the solutions given in (20) are not redundant. Since $(r, a) = 1$ and $b > s$, the solutions in (20) are not improper. Finally, since $x_1 = y_1 = 0$ and $(r, s) = 1$, the set of solutions given by (20) is not reducible. \square

Remark: Theorem 3 still holds if $\min(x, y) > 0$, provided we revise the definition of reducibility to require $w > 0$ and $z > 0$. For example, when $\min(x, y) > 0$, (20) becomes

$$(a, b, c, r, s, x_1, y_1, x_2, y_2, x_3, y_3) = (a_0, dA, dA^2(d-1)/h_1, d(dA-1)/h_1, (A-1)/h_1, j, 1, 2j, 2, (m+1)j, 3),$$

where $h_1 = \gcd(d(dA - 1), A - 1)$.

The definitions of A , B , m , and n in the proof of Theorem 3 show that any set of three solutions to (17) (for given a , b , c , r , s) corresponds to a unique solution (A, B, m, n) to the Goormaghtigh equation (19). On the other hand, each solution (A, B, m, n) to (19) corresponds to an infinite number of sets of three solutions to (17). However, it is not hard to show that each solution (A, B, m, n) to (19) corresponds to a unique set of three solutions to (17) which is not reducible, improper, or redundant.

The familiar Goormaghtigh conjecture says that, taking $A < B$, the only solutions (A, B, m, n) to (19) with $n > 2$ are given by $(A, B, m, n) = (2, 5, 5, 3)$ and $(2, 90, 13, 3)$. (See [6] and [13].) If this well-known conjecture is true, then the only two choices of $(a, b, c, r, s, x_1, y_1, x_2, y_2, x_3, y_3)$ which are not given by (20) and which give a set of three solutions to (17) which is not reducible, improper, or redundant are:

$$(2, 5, 3, 1, 1, 2, 0, 3, 1, 7, 3), (2, 90, 88, 89, 1, 0, 0, 1, 1, 13, 3) \quad (21).$$

If $\min(x, y) > 0$, (21) becomes $(2, 5, 15, 5, 1, 2, 1, 3, 2, 7, 4), (2, 90, 7920, 4005, 1, 1, 1, 2, 2, 14, 4)$.

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